VOLUME ESTIMATE ABOUT SHRINKERS

XU CHENG AND DETANG ZHOU

ABSTRACT. We derive a precise estimate on the volume growth of the level set of a potential function on a complete noncompact Riemannian manifold. As applications, we obtain the volume growth rate of a complete noncompact self-shrinker and a gradient shrinking Ricci soliton. We also prove the equivalence of weighted volume finiteness, polynomial volume growth and properness of an immersed self-shrinker in Euclidean space.

1. INTRODUCTION

In this paper, we estimate the volume of complete noncompact Riemannian manifolds without assuming curvature conditions. Let (M, g) be an n-dimensional complete noncompact Riemannian manifold on which there exists a nonnegative smooth function f satisfying "shrinking soliton" conditions.

Let Δ_f be the linear operator defined by

$$\Delta_f = \Delta - \langle \nabla f, \quad \rangle.$$

It is known that the operator Δ_f is symmetric with respect to the weighted measure $e^{-f} dv$ on the space of smooth function with compact support. Denote the level set of $2\sqrt{f}$, its volume and its weighted volume by

$$D_r = \{x \in M : 2\sqrt{f} < r\},$$

$$V(r) = \text{Vol}(D_r) = \int_{D_r} dv,$$

and

$$V_f(r) = \text{Vol}_f(D_r) = \int_{D_r} e^{-f} dv.$$

We get the volume growth estimate of the set D_r and the finiteness of the weighted volume of M in the following

Theorem 1.1. *(Thm.2.1) Let (M, g) be a complete noncompact Riemannain manifold. If f is a proper C^∞ function on M satisfying $|\nabla f|^2 \leq f$ on D_r for all r and*

$$(1) \qquad\qquad\qquad \Delta_f f + f \leq k$$

for some constant k, then M has finite weighted volume, that is,

$$V_f(M) = \int_M e^{-f} dv < +\infty$$

and

$$V(r) \leq Cr^{2k}$$

for $r \geq 1$, where C is a constant depending only on $\int_M e^{-f} dv$.

The classical volume comparison theorems in Riemannian geometry are one of the basic ingredients of the analysis on manifolds. It is usually done by assuming that M is an n-dimensional Riemannian manifold with Ricci curvature being bounded below. In our theorem, instead of assuming curvature conditions, we assume the existence of some functions which arise naturally in the study of gradient shrinking Ricci solitons of Ricci flow and self-shrinkers of mean curvature flow. Both gradient shrinking Ricci solitons and self-shrinkers are important in the singularity analysis of Ricci flow and mean curvature flow respectively ([10], [11], [12]) and recently have been studied very much (see [6], [7], survey papers [1], [2], etc). We observe both of them satisfy inequality (1) and so Theorem 1.1 can be applied to them. Besides, Theorem 1.1 is of independent interest. We only need to find a proper function on M satisfying the assumption of the theorem.

As the first application, we consider self-shrinkers for mean curvature flow (MCF) in $\mathbb{R}^{n+1}$. Recall that a one-parameter family of hypersurfaces $\Sigma_t \subset \mathbb{R}^{n+1}$ flows by mean curvature if

$$\partial_t x = -H\mathbf{n},$$

where $\mathbf{n}$ is the outward unit normal and H is the mean curvature. The flow Σ_t is said to be a self-shrinker if

$$\Sigma_t = \sqrt{-t}\Sigma_{-1}$$

for all $t < 0$. This is equivalent to that $\Sigma = \Sigma_{-1}$ satisfies the equation

$$(2) \qquad\qquad\qquad H = \frac{\langle x, \mathbf{n} \rangle}{2}.$$

We also call hypersurface Σ a self-shrinker in $\mathbb{R}^{n+1}$. We obtain

Theorem 1.2. *Let Σ^n be a complete noncompact properly immersed self-shrinker in Euclidian space $\mathbb{R}^{n+1}$ with nonnegative constant $\beta \leq \inf H^2$. Then, there is a positive constant C such that for $r \geq 1$*

$$\mathrm{Vol}(B_r(0) \cap \Sigma) \leq Cr^{n-2\beta}.$$

The upper bounds of the volume in Theorem 1.2 are optimal because $\mathbb{R}^n$ and cylinder self-shrinker $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ have the corresponding volume growth rates. In [8], Ding and Xin proved the Euclidean volume growth of a complete noncompact properly immersed self-shrinker. Theorem 1.2 implies their result.

Further, in Section 4 we study the relation among weighted volume finiteness, polynomial volume growth and properness of a self-shinker. In the works of Ecker-Huisken [9] and Colding-Minicozzi [6] the polynomial volume growth plays an important role in studying self-shrinker. It is satisfied when the self-shrinker is a time-slice of a blow up. In general, for a complete noncompact self-shrinker Σ, we show in Theorem 1.3 that the following statements are equivalent,

Theorem 1.3. *For any complete immersed self-shrinker Σ^n in $\mathbb{R}^{n+1}$, the following statements are equivalent,*

   (i) *Σ is proper;*
  (ii) *Σ has Euclidean volume growth;*
 (iii) *Σ has polynomial volume growth;*
 (iv) *Σ has finite weighted volume, that is, $\int_{\Sigma} e^{-\frac{|x|^2}{4}} dv < \infty$.*

Theorem 1.2 and 1.3 are the consequences of the corresponding general results of self-shrinkers in $\mathbb{R}^{n+p}, p \geq 1$ (see Theorem 3.1 and 4.1).

Using Theorem 1.3, we can rephrase the compactness theorem of Colding-Minicozzi [7] as the following

Theorem 1.4. *(Colding-Minicozzi [7]) Given an integer $g \geq 0$ and a constant $A > 0$, the set*

$$
\mathcal{S}(g, A) := \left\{ \begin{array}{c} \Sigma : \Sigma \text{ is a complete smooth embedded self-shrinker in } \mathbb{R}^3 \text{ with} \\[2mm] \text{genus at most } g \text{ and } \int_{\Sigma} e^{-\frac{|x|^2}{4}} dv \leq A \end{array} \right\}
$$

is compact with respect to the topology of C^m convergence on compact subsets for any $m \geq 2$.

The second application of Theorem 1.1 is about gradient shrinking solitons. Recall that a complete smooth Riemannian manifold (M^n, g) is called a *normalized gradient shrinking Ricci soliton* if there is a smooth function f on M^n such that the Ricci tensor R_{ij} of the metric g is given by

$$
(3) \qquad\qquad R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}.
$$

It is known that the scalar curvature R of the metric g is nonnegative [5]. We get

Theorem 1.5. *(Thm.5.1) Let (M^n, g, f) be a complete noncompact gradient shrinking solition with f satisfying (3) and $\beta = \inf R$. Then there exists a constant $C > 0$ such that*

$$
\mathrm{Vol}(B_r(x_0)) \leq C r^{n-2\beta}
$$

for $r \geq 1$, where C is a constant depending on x_0 and $\mathrm{Vol}(B_r(x_0))$ denotes the volume of the geodesic ball of M of radius r centered at $x_0 \in M$.

Since the scalar curvature R is nonnegative ([5]), Theorem 1.5 implies the estimate of Euclidean volume growth which has been obtained by Cao and the second author in [5]. The volume upper bounds of the geodesic ball $B_r(x_0)$ in Theorem 1.5 are sharp by the examples like the Gaussian shrinker and cylinder shrinking solitons. It has been proved (Cao-Chen-Zhu [3]), that *any 3-dimensional complete noncompact non-flat shrinking gradient soliton is necessarily the round cylinder $\mathbb{S}^2 \times \mathbb{R}$ or one of its $\mathbb{Z}_2$ quotients.* But for higher dimensions, one has no classification of complete noncompact gradient shrinking solitons.

2. Volume growth estimate

In this section, we will prove Theorem 1.1. The first part of our argument is similar to the proof of Ding and Xin in the case of self-shrinkers. We have the following

Theorem 2.1. *(Thm.1.1) Let (M, g) be a complete noncompact Riemannain manifold. Let f be a proper C^∞ function on M. If $|\nabla f|^2 \leq f$ on D_r for all r and*

$$(4) \qquad\qquad \Delta_f f + f \leq k$$

for some constant k, then the weighted volume $\int_M e^{-f} dv < +\infty$ and

$$(5) \qquad\qquad V(r) \leq Cr^{2k}$$

for $r \geq 1$, where C is a constant depending only on $\int_M e^{-f} dv$.

Proof. Since f is proper, it is well defined the integral

$$I(t) = \frac{1}{t^k} \int_{\bar{D}_r} e^{-\frac{f}{t}} dv, \quad t > 0.$$

$$(6) \qquad\qquad I'(t) = t^{-k-1} \int_{\bar{D}_r} (-k + \frac{f}{t}) e^{-\frac{f}{t}} dv.$$

On the other hand,

$$\begin{aligned}
\int_{\bar{D}_r} \operatorname{div}(e^{-\frac{f}{t}} \nabla f) dv &= \int_{\bar{D}_r} e^{-\frac{f}{t}} (\Delta f - \frac{1}{t} |\nabla f|^2) dv \\
&\leq \int_{\bar{D}_r} e^{-\frac{f}{t}} (|\nabla f|^2 - f + k - \frac{1}{t} |\nabla f|^2) dv \\
&= \int_{\bar{D}_r} e^{-\frac{f}{t}} (\frac{t-1}{t} |\nabla f|^2 - f + k) dv \\
&\leq \int_{\bar{D}_r} e^{-\frac{f}{t}} (\frac{t-1}{t} f - f + k) dv, \quad t \geq 1 \\
(7) \qquad &= \int_{\bar{D}_r} e^{-\frac{f}{t}} (-\frac{1}{t} f + k) dv,
\end{aligned}$$

Substituting (7) into (6) gives

$$I'(t) \le -t^{-k-1} \int_{\bar{D}_r} \operatorname{div}(e^{-\frac{f}{t}} \nabla f) dv.$$

At the regular value r of f and for $t \ge 1$, by Stokes' Theorem, we have

$$I'(t) \le -t^{-k-1} \int_{\partial D_r} \left\langle e^{-\frac{f}{t}} \nabla f, \frac{\nabla f}{|\nabla f|} \right\rangle dv$$
$$= -t^{-k-1} \int_{\partial D_r} e^{-\frac{f}{t}} |\nabla f| dv \le 0.$$

Integrating $I'(t)$ over t from 1 to $r^2 > 1$, we get

$$(8) \qquad r^{-2k} \int_{\bar{D}_r} e^{-\frac{f}{r^2}} dv \le \int_{\bar{D}_r} e^{-f} dv.$$

Since the integrals in (8) are upper semi-continuous for all r, (8) holds for all $r \ge 1$. Note $2\sqrt{f} \le r$ over $\bar{D}_r$. (8) implies for all $r \ge 1$

$$e^{-\frac{1}{4}} r^{-2k} \int_{\bar{D}_r} dv \le r^{-2k} \int_{\bar{D}_r} e^{-\frac{f}{r^2}} dv$$
$$(9) \qquad\qquad\qquad \le \int_{\bar{D}_r} e^{-f} dv.$$

Note that

$$\int_{\bar{D}_r} e^{-f} dv - \int_{\bar{D}_{r-1}} e^{-f} dv = \int_{\bar{D}_r \setminus \bar{D}_{r-1}} e^{-f} dv$$
$$\le e^{-\frac{(r-1)^2}{4}} \int_{\bar{D}_r \setminus \bar{D}_{r-1}} dv$$
$$(10) \qquad\qquad\qquad \le e^{-\frac{(r-1)^2}{4}} \int_{\bar{D}_r} dv.$$

By (9) and (10), we have

$$(11) \qquad \int_{\bar{D}_r} e^{-f} dv - \int_{\bar{D}_{r-1}} e^{-f} dv \le e^{\frac{1}{4}} r^{2k} e^{-\frac{(r-1)^2}{4}} \int_{\bar{D}_r} e^{-f} dv.$$

We can choose a constant r_0 such that $e^{\frac{1}{4}} r^{2k} e^{-\frac{(r-1)^2}{4}} < e^{-r}$, for $r > r_0$. Then (11) implies

$$(12) \qquad \int_{\bar{D}_r} e^{-f} dv \le \frac{1}{1 - e^{-r}} \int_{\bar{D}_{r-1}} e^{-f} dv.$$

Then for any positive integer N, we have

$$(13) \qquad \int_{\bar{D}_{r+N}} e^{-f} dv \le \left(\prod_{i=0}^{N} \frac{1}{1 - e^{-(r+i)}} \right) \int_{\bar{D}_{r-1}} e^{-f} dv < \infty.$$

This implies that $\int_M e^{-f} dv < +\infty$. Moreover by (9),

$$e^{-\frac{1}{4}r^{-2k}} \int_{\bar{D}_r} dv \le \int_{\bar{D}_r} e^{-f} dv \le \int_M e^{-f} dv.$$

So for all $r \ge 1$, $V(r) \le Cr^{2k}$.   □

3. APPLICATION TO SELF-SHRINKERS

In this section, we discuss the volume of self-shrinkers of mean curvature flow by using Thoerem 2.1. Let Σ be an n-dimensional submanifold isometrically immersed in $\mathbb{R}^{n+p}$. Let $A(V, W) = \overline{\nabla}_V W - \nabla_V W, V, W \in T\Sigma$ denote the second fundamental form and $\mathbf{H}$ denote the mean curvature vector of Σ given by $\mathbf{H} = \sum_{i=1}^n A(e_i, e_i)$, where $\{e_i\}$ is a local orthonormal frame field of Σ. Σ is said to be a self-shrinker in $\mathbb{R}^{n+p}$ if it satisfies

$$(14) \qquad\qquad\qquad \mathbf{H} = -\frac{x^N}{2},$$

where x denotes the position vector, $(\cdot)^N$ denotes the orthogonal projection into the normal bundle $N\Sigma$ of Σ.

Let $f(x) = \frac{|x|^2}{4}$. Then $\bar{\nabla} f = \frac{x}{2}, |\bar{\nabla} f|^2 = \frac{|x|^2}{4}$,

$$(15) \qquad f - |\nabla f|^2 = f - |\bar{\nabla} f|^2 + |(\bar{\nabla} f)^N|^2 = \frac{|x^N|^2}{4} = |\mathbf{H}|^2 \ge 0,$$

$$
\begin{aligned}
\Delta_\Sigma f &= \frac{1}{4} \sum_{i=1}^n \bar{D}^2(|x|^2)(e_i, e_i) + \frac{1}{4}\langle \mathbf{H}, \bar{\nabla}|x|^2\rangle \\
(16) \qquad\quad &= \frac{n}{2} - \frac{1}{4}\langle x^N, x\rangle^2 \\
&= \frac{n}{2} - f + |\nabla f|^2.
\end{aligned}
$$

Denote by $B_r(0) = \{x \in \mathbb{R}^{n+1}; |x| < r\}$ the ball in $\mathbb{R}^{n+1}$ of radius r centered at the origin.

Theorem 3.1. *Let Σ^n be a complete noncompact properly immersed self-shrinker in Euclidian space $\mathbb{R}^{n+p}$ with nonnegative constant $\beta \le \inf |\mathbf{H}|^2$. Then, there is a positive constant C such that for $r \ge 1$*

$$\text{Vol}(B_r(0) \cap \Sigma) \le Cr^{n-2\beta}.$$

Proof. By (15), $\inf\{f - |\nabla f|^2\} = \inf |\mathbf{H}|^2 \ge \beta \ge 0$. Take $\bar{f} = f - \beta$. Then

$$\bar{f} - |\bar{\nabla} f|^2 = f - |\nabla f|^2 - \beta \ge 0.$$

$$\Delta_\Sigma \bar{f} = \frac{n}{2} - \beta - \bar{f} + |\nabla \bar{f}|^2.$$

The properness of the immersion of Σ is the properness of f on Σ and hence $\bar{f}$ is proper. The above implies $\bar{f}$ satisfies the assumption of Theorem 2.1. Denote

$$D_r = \{x \in \Sigma : 2\sqrt{\bar{f}} < r\}$$

Then

$$D_r = \{x \in \Sigma : |x| < \sqrt{r^2 + 4\beta}\}.$$

By Theorem 2.1,

$$\text{Vol}(D_r) \leq Cr^{n-2\beta},$$

that is

$$\text{Vol}(B_{\sqrt{r^2+4\beta}}(0) \cap \Sigma) \leq Cr^{n-2\beta}.$$

So

$$\text{Vol}(B_r(0) \cap \Sigma) \leq Cr^{n-2\beta}.$$

$$\square$$

In particular, if Σ^n be a self-shrinker in $\mathbb{R}^{n+1}$. Since $\mathbf{H} = -H\mathbf{n}$, where $H = -\langle \nabla_{e_i} e_i, \mathbf{n} \rangle$ and $\mathbf{n}$ denote the mean curvature and the outward unit normal respectively, Σ satisfies

$$(17) \qquad\qquad H = \frac{\langle x, \mathbf{n} \rangle}{2},$$

So we have Theorem 1.2.

Remark 3.1. Take cylinder self-shrinker $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$, $0 \leq k \leq n$. $H = \frac{\sqrt{k}}{\sqrt{2}}$, $n - 2\beta = n - k$. Theorem 3.1 implies $\text{Vol}(B_r(0) \cap \Sigma) \leq Cr^{n-k}$, which is sharp. The constant C in theorem depends only on the integral $\int_\Sigma e^{-\frac{|x|^2}{4}} dv < \infty$ and β explicitly as seen in the proof. Take $\beta = 0$ in Theorem 3.1. We have the Euclidean growth upper bound proved by Ding and Xin [8].

4. Properness, polynomial volume growth and finite weighted volume for self-shrinkers

In this section, we discuss the equivalence of properness, polynomial volume growth and finite weighted volume for complete self-shrinkers. If Σ is an n-dimensional submanifold in $\mathbb{R}^{n+p}$, $p \geq 1$, Σ is said to have polynomial volume growth if there exist constants C and d so that for all $r \geq 1$

$$(18) \qquad\qquad \text{Vol}(B_r(0) \cap \Sigma) \leq Cr^d.$$

When $d = n$ in (18), Σ is said to has Euclidean volume growth. We get the following

Theorem 4.1. *For any complete n-dimensional immersed self-shrinker Σ^n in $\mathbb{R}^{n+p}$, $p \geq 1$, the following statements are equivalent,*

(i) Σ *is proper;*
(ii) Σ *has Euclidean volume growth;*
(iii) Σ *has polynomial volume growth;*
(iv) Σ *has finite weighted volume, that is,* $\int_\Sigma e^{-\frac{|x|^2}{4}} dv < \infty$.

Proof. $(i) \Rightarrow (ii)$ has been proved in Theorem 3.1. $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are obvious. So we only need to prove that (iv) implies (i). By contrary, if Σ is not properly immersed, there exists a number $R > 0$ such that the pre-image $E \subseteq \Sigma$ of $\bar{B}_R$ is not compact in Σ. Then for a positive constant $a > 0$, there exists a sequence $\{p_k\}$ of points in E with $d_\Sigma(p_k, p_j) \geq a > 0$ for any $k \neq j$, where d_Σ denotes the intrinsic distance in Σ. Denote by $B_r^\Sigma(p)$ the geodesic ball of Σ of radius r centered at $p \in \Sigma$. So $B_{\frac{a}{2}}^\Sigma(p_k) \cap B_{\frac{a}{2}}^\Sigma(p_j) = \emptyset$ for any $k \neq j$. Choose $a < 2R$. Then $B_{\frac{a}{2}}^\Sigma(p_k) \subset B_{2R}$. Since

$$(19) \qquad \mathbf{H} = -\frac{x^N}{2},$$

for any $p \in \Sigma \cap B_{2R}$,

$$(20) \qquad |\mathbf{H}|(p) \leq \frac{|x|}{2} \leq \frac{2R}{2} = R.$$

We claim that there is a constant C depending only on R such that $\mathrm{vol}(B_{\frac{a}{2}}^\Sigma(p_k)) \geq Ca^n$ when $0 < a \leq \min\{2R, \frac{n}{2R}\}$ is suitably small.

In fact for $p \in B_{\frac{a}{2}}^\Sigma(p_k))$, the extrinsic distance function $r_k(p) := \mathrm{d}(p, p_k)$ from p_k satisfies

$$\Delta_\Sigma r_k^2 = 2n + 2\left\langle \mathbf{H}, r_k \bar{\nabla} r_k \right\rangle$$
$$\geq 2n - 2|\mathbf{H}|r_k$$
$$(21) \qquad \geq 2n - 2r_k R.$$

By $a \leq \frac{n}{2R}$, we have for $0 < \mu \leq \frac{a}{2}$

$$\int_{B_\mu^\Sigma(p_k)} (2n - 2Rr_k)dv \leq \int_{B_\mu^\Sigma(p_k)} \Delta_\Sigma r_k^2 dv$$
$$= \int_{\partial B_\mu^\Sigma(p_k)} \left\langle \nabla^\Sigma r_k^2, \nu \right\rangle dv$$
$$(22) \qquad \leq 2\mu A(\mu),$$

where ν denotes the outward unit normal vector of $\partial B_\mu^\Sigma(p_k)$ and $A(\mu)$ denotes the area of $\partial B_\mu^\Sigma(p_k)$. Then using co-area formula in (22), we get

$$(23) \qquad \int_0^\mu (n - Rs)A(s)ds \leq \int_0^\mu \int_{d_\Sigma(p,p_k)=s} (n - Rr_k)d\sigma \leq \mu A(\mu).$$

This implies

$$(n - R\mu)V(\mu) \leq \mu V'(\mu),$$

where $V(\mu)$ denotes the volume of $B_\mu^\Sigma(p_k)$. So

$$(24) \qquad \frac{V'(\mu)}{V(\mu)} \geq \frac{n}{\mu} - R$$

Integrating (24) from $\epsilon > 0$ to μ, we have

$$\frac{V(\mu)}{V(\epsilon)} \geq (\frac{\mu}{\epsilon})^n e^{-R(\mu - \epsilon)}$$

Since $\lim\limits_{s \to 0^+} \dfrac{V(s)}{s^n} = \omega_n$,

(25) $$V(\mu) \geq \omega_n \mu^n e^{-R\mu}.$$

The claim follows from (25). Now we have

(26)
$$\int_\Sigma e^{-\frac{|x|^2}{4}} dv \geq \sum_{k=1}^\infty \int_{B_{\frac{a}{2}}^\Sigma (p_k))} e^{-\frac{|x|^2}{4}} dv$$

$$\geq e^{-R^2} \sum_{k=1}^\infty \int_{B_{\frac{a}{2}}^\Sigma (p_k)} dv = +\infty.$$

It is a contradiction with the hypothesis of the finiteness of the weighted volume. $\qquad\square$

Take $p = 1$ in Theorem 4.1, we get Theorem 1.3.

Remark 4.1. It was asked by Huai-Dong Cao if a complete self-shrinker automatically has polynomial volume growth. So Theorem 4.1 answers this question. If there is any example of nonproperly immersed self-shrinker, it must have infinite weighted volume. We do not know any such examples.

5. Application to gradient shrinking Solitons

In this section, we consider gradient shrinking solitons of Ricci flows. A complete n-dimensional smooth Riemannian manifold (M, g) is called a gradient shrinking Ricci soliton if there exists a smooth function f on M^n such that the Ricci tensor R_{ij} of the metric g is given by

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}$$

for some positive constant $\rho > 0$. By scaling g_{ij} one can normalize $\rho = \frac{1}{2}$ so that

(27) $$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}.$$

Take trace in (27), we have

$$R + \Delta f = \frac{n}{2}.$$

It was proved by Hamilton [10] that there is a constant C_0 such that the scalar curvature R and f satisfy

(28) $$R + |\nabla f|^2 - f = C_0.$$

Since B.L. Chen [5] proved a complete gradient shrinking soliton has non-negative scalar curvature (cf. Proposition 5.5 [1]), we can choose a new f, still denoted by f, such that C_0 is equal to the infimum of R. Denote by

$\beta = \inf R$. Hence for this f, $0 \leq |\nabla f|^2 \leq f$ and thus (M, g) is a gradient shrinking Ricci soliton with nonnegative potential function f and

$$(29) \qquad \Delta f - |\nabla f|^2 + f = \frac{n}{2} - \beta.$$

Using Theorem 2.1, we get the following

Lemma 5.1. *Let (M^n, g, f) be a complete noncompact gradient shrinking solition with f satisfying (27) and (29) with $\beta = \inf R$. Then $\int_M e^{-f} dv < \infty$ and*

$$V(r) \leq Cr^{n-2\beta}$$

for $r \geq 1$, where C is a constant depending only on $\int_M e^{-f} dv$.

Proof. It suffices to show the properness of f. Take $\bar{f} = f + \beta$. In [4], it is proved that the potential function $\bar{f}$ satisfies for r sufficiently large

$$(30) \qquad \frac{1}{4}(r(x) - c_1)^2 \leq \bar{f}(x) \leq \frac{1}{4}(r(x) + c_2)^2,$$

where $r(x) = d(x_0, x)$ is the distance function from some fixed point $x_0 \in M$, c_1 and c_2 are positive constants depending only on n and the geometry of g_{ij} on the unit ball $B_{x_0}(1)$. Hence $\bar{f}$ is proper and thus f is also. $\qquad \square$

Theorem 5.1. *(Thm.1.5) Let (M^n, g, f) be a complete noncompact gradient shrinking solition with f satisfying (27) and $\beta = \inf R$. Then there exists a constant $C > 0$ such that*

$$\mathrm{Vol}(B_r(x_0)) \leq Cr^{n-2\beta}$$

for $r \geq 1$, where C is a constant depending on x_0 and $\mathrm{Vol}(B_r(x_0))$ denotes the volume of the geodesic ball of M of radius r centered at $x_0 \in M$.

Proof. Take $\tilde{f} = f + \beta - C_0$. Then $\tilde{f}$ satisfies Lemma 5.1. So let $D_r = \{x \in M : 2\sqrt{\tilde{f}} < r\}$. Then

$$V(D_r) \leq Cr^{n-2\beta}$$

for $r \geq 1$. By (30), it is easy to know $B_r(x_0) \subseteq D_{r+c}$ for some constant c and sufficiently large r. $\qquad \square$

Remark 5.1. The volume upper bounds of the level set D_r and the geodesic ball $B_r(x_0)$ in Theorem 5.1 are optimal. For example the Gaussian shrinker, namely the flat Euclidean space $(\mathbb{R}^n, g_0)$ with the potential function $f = |x|^2/4$ and cylinder shrinking solitons $(S^{n-k} \times \mathbb{R}^k, g)$, $k \geq 1$, where $g = 2(n-k-1)g_{S^{n-k}} + g_{\mathbb{R}^k}$ and $f(\theta, x) = \frac{|x|^2}{4}, \theta \in S^{n-k}, x \in \mathbb{R}^k$.

Remark 5.2. Since $\beta = \inf R \geq 0$, Theorem 5.1 implies the Euclidean volume growth estimate of Cao and the second author [4]. The method of the proof presented in Lemma 5.1 is different .

Remark 5.3. By [4] the equation of shrinking Ricci soliton gives the estimate of potential function, we can use the finiteness of weighted volume to conclude that the fundamental group $\pi_1(\Sigma)$ of Σ is finite, which was proved by W. Wylie [13]. But for a self-shrinker, the potential function comes from the restriction of metric of ambient space. We cannot use it to study the fundamental group of self-shrinker.

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Instituto de Matematica, Universidade Federal Fluminense, Niterói, RJ 24020, Brazil

E-mail address: xcheng@impa.br

E-mail address: zhou@impa.br